THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 I/J University Mathematics 2015-2016
Suggested Solution to Problem Set 5

1. Note that

$$\begin{aligned}\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^+}\frac{\cos(0+h)-\cos(0)}{h}\\ &= \lim_{h\to 0^+}\frac{\cos h-1}{h}\\ &= \lim_{h\to 0^+} -\frac{1}{2}\left(\frac{\sin^2(\frac{h}{2})}{(\frac{h}{2})^2}\right)h\\ &= (-\frac{1}{2})(1)^2(0)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^+}\frac{1-\cos(0)}{h}\\ &= \lim_{h\to 0^-}\frac{0}{h}\\ &= \lim_{h\to 0^-} 0\\ &= 0\end{aligned}$$

$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} = 0$ and so $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ exists which means $f(x)$ is differentiable at $x=0$. In particular, $f'(0)=0$.

2. Note that

$$\begin{aligned}\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^+}\frac{h^{2/3}}{h}\\ &= \lim_{h\to 0^+}\frac{1}{h^{1/3}}\end{aligned}$$

which goes to positive infinity and it means $\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}$ does not exist. Therefore, $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ does not exist and $f(x)$ is not differentiable at $x=0$.

3. Note that $f(x)$ is differentiable at $x=1$, so

$$\begin{aligned}\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}\\ \lim_{h\to 0^+}\frac{(a(1+h)+b)-1}{h} &= \lim_{h\to 0^-}\frac{(1+h)^3-1}{h}\\ \lim_{h\to 0^+}\frac{ah+(a+b-1)}{h} &= \lim_{h\to 0^-} 3+3h+h^2\\ \lim_{h\to 0^+}\frac{ah+(a+b-1)}{h} &= 3\end{aligned}$$

First of all, the limit on the left hand side exists if and only if $a+b-1=0$. When $a+b-1=0$, the limit on the left hand side is a which has to be 3 by the equation. Therefore, we have $a=3$ and $b=-2$.

4. (a) $f(0) = \lim_{n\to\infty} \frac{a(n^0 - n^{-0})}{n^0 + n^{-0}} = \lim_{n\to\infty} 0 = 0.$

   (b) If $x > 0$,

$$\begin{aligned} f(x) &= \lim_{n\to\infty} \frac{a(n^x - n^{-x})}{n^x + n^{-x}} \\ &= \lim_{n\to\infty} \frac{a(1 - n^{-2x})}{1 + n^{-2x}} \\ &= a \end{aligned}$$

   If $x < 0$,

$$\begin{aligned} f(x) &= \lim_{n\to\infty} \frac{a(n^x - n^{-x})}{n^x + n^{-x}} \\ &= \lim_{n\to\infty} \frac{a(n^{2x} - 1)}{n^{2x} + 1} \\ &= -a \end{aligned}$$

   (c) If $f(x)$ is continuous at $x = 0$, we have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0).$$

   Therefore, we have $a = 0$.

5. Let $x \in \mathbb{R}$. We have

$$\begin{aligned} \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} &= \lim_{h\to 0} \frac{(g(x)f(h) + f(x)g(h)) - g(x)}{h} \\ &= \lim_{h\to 0} g(x)\left(\frac{f(h)-1}{h}\right) + f(x)\left(\frac{g(h)}{h}\right) \\ &= \lim_{h\to 0} g(x)\left(\frac{f(h)-f(0)}{h}\right) + f(x)\left(\frac{g(h)-g(0)}{h}\right) \\ &= g(x)f'(0) + f(x)g'(0) \\ &= f(x) \end{aligned}$$

Therefore, $g'(x) = f(x)$.